Technical Note 1971-32

A New Geometrical Theorem Discovered with the Aid of a Computer

W. L. Black
H. C. Howland
B. Howland

21 June 1971

Prepared under Electronic Systems Division Contract F19628-70-C-0230 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

A NEW GEOMETRICAL THEOREM DISCOVERED WITH THE AID OF A COMPUTER

W. L. BLACK

Division 6

H. C. HOWLAND

Cornell University
Division of Biological Sciences
Ithaca, New York 14850

B. HOWLAND

Group 63

TECHNICAL NOTE 1971-32

21 JUNE 1971

LEXINGTON MASSACHUSETTS

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the Department of the Air Force under Contract F19628-70-C-0230.

ABSTRACT

In the course of a computer study of a new form of ball bearing, a curious invariance was noted. This led to a new theorem in the geometry of circles. A proof for this theorem, together with a useful lemma, is the subject of this Technical Note.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col., USAF
Chief, Lincoln Laboratory Project Office

A NEW GEOMETRICAL THEOREM DISCOVERED WITH THE AID OF A COMPUTER

Let Γ and $\tilde{\Gamma}$ be two circles in euclidean 3-space, R^3. Suppose there is a number x such that: (1) every point on either circle is distance x from exactly two points on the other circle. We can then select a point Z_1 on Γ and draw a zig-zag line between the two circles as follows:

$$
\begin{array}{llllll}
\text{Select} & \tilde{Z}_1 & \text{on} & \tilde{\Gamma} & \text{with} & |\tilde{Z}_1 - Z_1| = x \\
\text{Select} & Z_2 \neq Z_1 & \text{on} & \Gamma & \text{with} & |Z_2 - \tilde{Z}_1| = x \\
\text{Select} & \tilde{Z}_2 \neq \tilde{Z}_1 & \text{on} & \tilde{\Gamma} & \text{with} & |\tilde{Z}_2 - Z_2| = x \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
\text{Select} & Z_{n+1} \neq Z_n & \text{on} & \Gamma & \text{with} & |Z_{n+1} - \tilde{Z}_n| = x \\
\text{Select} & \tilde{Z}_{n+1} \neq \tilde{Z}_n & \text{on} & \tilde{\Gamma} & \text{with} & |\tilde{Z}_{n+1} - Z_{n+1}| = x \quad .
\end{array}
$$

It may happen (illustrated for the case $n = 3$ in Fig. 1) that $Z_{n+1} = Z_1$. We show that if this occurs, the zig-zag line can be started at <u>any</u> point on Γ and it will still close.

This remarkable fact was observed while performing certain calculations about ball bearings on a computer[1]. The theorem bears a superficial resemblance to Steiner's Porism[2] but cannot be proved the same way.

Condition (1) is not as formidable as it may appear. If both circles lie in the same plane, with radii r and $\bar{r}$ and with centers separated by δ, elementary calculus shows that (1) is equivalent to the two inequalities:

$$|r - \bar{r}| < x - \delta$$

$$x + \delta < r + \bar{r}$$

Thus suitable x's will exist provided the smaller circle encloses the center of the larger one.

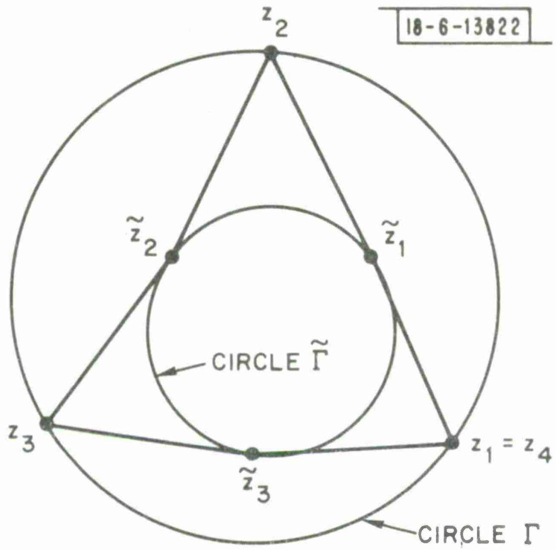

Fig. 1. Each straight line has length x, n = 3.

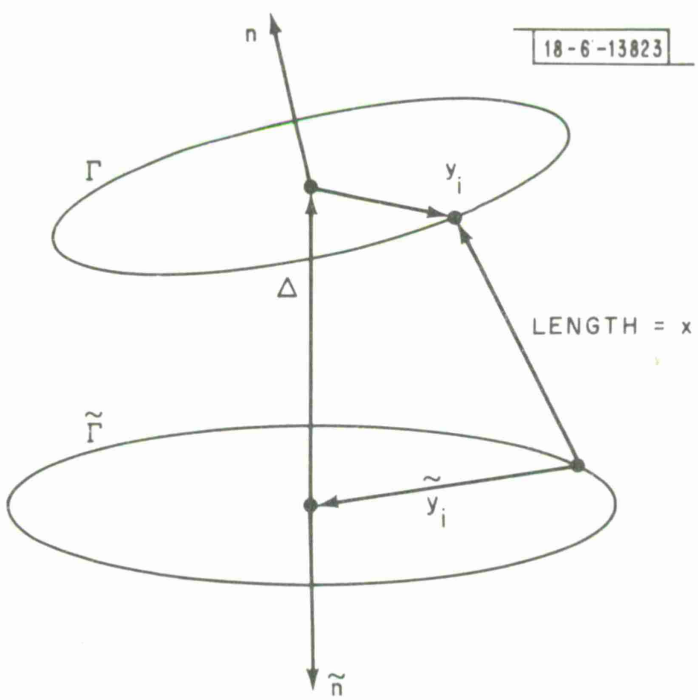

Fig. 2. Illustrating the notation used in the computation of f.

The proof proceeds as follows: Suppose for some choice of Z_1, say $Z_1 = p$, $Z_{n+1} = Z_1$. We parametrize Γ by s, the directed arc length measured from p. Thus we can view Z_1, and thus Z_{n+1}, and ultimately t, the arc length from p to Z_{n+1} as functions of s. Below we show that there is a smooth function f of s and t with properties:

$$\text{(i.)}\ f(s,s) = 1 \text{ for all } s$$

$$\text{(ii.)}\ f(s,t) = \frac{dt}{ds}$$

Application of a well known uniqueness theorem[3] assures us that this ordinary differential equation:

$$\frac{dt}{ds} = f(s,t)$$

$$t(o) = 0$$

has only one solution. By (i), $t(s) = s$ is the solution. But this implies that $Z_{n+1} = Z_1$ for all Z_1.

In constructing f we will need the following lemma:

<u>Lemma</u>: Let B, n, Z_1 and Z_2 be vectors in R^3, satisfying:

$$\text{(a.)}\ |y_1| = |y_2|$$

$$\text{(b.)}\ |B + y_1| = |B + y_2|$$

$$\text{(c.)}\ n \cdot y_1 = n \cdot y_2$$

$$\text{(d.)}\ y_1 \neq y_2$$

Then:

$$B \times n \cdot (y_1 + y_2) = 0 \quad .$$

<u>Proof</u>: If B and n are dependent the result is obvious. If not, by squaring (b) and (a) and taking the difference we obtain:

$$\text{(e.)}\ B \cdot y_1 = B \cdot y_2$$

In conjunction with (c), (e) shows that y_1 and y_2 have the same orthogonal projection on the plane spanned by B and n. By (a) and (d) the components of y_1 and y_2 normal to this plane must be equal and opposite, so $y_1 + y_2$ lies in the B,n plane. From this the conclusion is evident.

We can now compute f. We will use the following notation:

n is the unit normal to circle Γ

$\tilde{n}$ " " " " $\tilde{\Gamma}$

y_i is the vector from the center of Γ to Z_i

$\tilde{y}_i$ is the <u>negative</u> of the vector from the center of $\tilde{\Gamma}$ to $\tilde{Z}_i$.

Δ is the vector from the center of $\tilde{\Gamma}$ to the center of Γ .

Figure two illustrates this notation. Imagine a slight motion of y_1 along the circle. Since dy_1 is perpendicular to both n and y_1 we can write:

$$dy_1 = n \times \frac{y_1}{|y_1|} |dy_1| \tag{2}$$

As y_1 moves, $\tilde{y}_1$ also must move, keeping

$$|\Delta + y_1 + \tilde{y}_1| = x \tag{3}$$

Squaring (3) and differentiating gives:

$$(\Delta + y_1 + \tilde{y}_1) \cdot (dy_1 + d\tilde{y}_1) = 0 \tag{4}$$

Substituting into (4), (2) and (5) where:

$$d\tilde{y}_1 = \tilde{n} \times \frac{\tilde{y}_1}{|y_1|} |d\tilde{y}_1| \tag{5}$$

gives:

$$(\Delta + \tilde{y}_1) \cdot n \times \frac{y_1}{|y_1|} |dy_1| + (\Delta + y_1) \cdot \tilde{n} \times \frac{\tilde{y}_1}{|\tilde{y}_1|} |d\tilde{y}_1| = 0 \quad . \tag{6}$$

Thus:

$$\frac{|d\tilde{y}_1|}{|dy_1|} = - \frac{|\tilde{y}_1| \; (\Delta + \tilde{y}_1) \cdot n \times y_1}{|y_1| \; (\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} \quad . \tag{7}$$

Similarly,

$$\frac{|dy_2|}{|d\tilde{y}_1|} = - \frac{|y_2| \; (\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_1}{|\tilde{y}_1| \; (\Delta + \tilde{y}_1) \cdot n \times y_2} \tag{8}$$

Multiplying (7) by (8) and using the lemma in the forms:

$$(\Delta + \tilde{y}_1) \cdot n \times y_1 = - (\Delta + \tilde{y}_1) \cdot n \times y_2$$

$$(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_1 = - (\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2$$

gives:

$$\frac{|dy_2|}{|dy_1|} = \frac{(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} \quad . \tag{9}$$

Multiplying n similar expressions gives:

$$\frac{dt}{ds} = \frac{|dy_{n+1}|}{|dy_1|} = \frac{(\Delta + y_{n+1}) \cdot \tilde{n} \times \tilde{y}_{n+1}}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} = f(s,t)$$

Clearly if $Z_{n+1} = Z_1$, then $y_{n+1} = y_1$ and $\tilde{y}_{n+1} = \tilde{y}_1$, whence:

$$f(s,s) = 1 \quad .$$

This completes the proof.

REFERENCES

1. Quarterly Progress Report No. 101, Research Laboratory of Electronics, Massachusetts Institute of Technology, p. 268 (15 April 1971).

2. H. S. M. Coxeter and M. C. Greitzer, Geometry Revisited, Random House, pp. 124, 125 (1967).

3. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, p. 10 (1955).

ACKNOWLEDGMENT

We wish to thank Mr. Fred Zimnoch for his programming assistance.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA – R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Lincoln Laboratory, M.I.T.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None
3. REPORT TITLE A New Geometrical Theorem Discovered with the Aid of a Computer	
4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)* Technical Note	
5. AUTHOR(S) *(Last name, first name, initial)* Black, William L. Howland, Howard C. Howland, Bradford	

6. REPORT DATE 21 June 1971	7a. TOTAL NO. OF PAGES 12	7b. NO. OF REFS 3
8a. CONTRACT OR GRANT NO. F19628-70-C-0230 b. PROJECT NO. 649L c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Note 1971-32 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-71-192	
10. AVAILABILITY/LIMITATION NOTICES Approved for public release; distribution unlimited.		
11. SUPPLEMENTARY NOTES None	12. SPONSORING MILITARY ACTIVITY Air Force Systems Command, USAF	

13. ABSTRACT

In the course of a computer study of a new form of ball bearing, a curious invariance was noted. This led us to a new theorem in the geometry of circles. A proof for this theorem, together with a useful lemma, is the subject of this Technical Note.

14. KEY WORDS

geometry of circles
theorem
lemma
computer study

UNCLASSIFIED

Security Classification